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Isabelle Gruais * **and Dan Poliševski** **

Abstract. We study the homogenization of a diffusion process which takes place in a binary structure formed by an ambiental connected phase surrounding a suspension of very small spheres distributed in an ε-periodic network. We consider the critical radius case with finite diffusivities in both phases. The asymptotic distribution of the concentration is determined, as $\varepsilon \to 0$, assuming that the suspension has mass of unity order and vanishing volume. It appears that the ambiental macroscopic concentration is satisfying a Volterra integro-differential equation and it is defining straightly the macroscopic concentration associated to the suspension.

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1 Introduction

The present study reveals the basic mechanism which governs diffusion in a binary structure, formed by an ambiental connected phase surrounding an ε-periodic suspension of small particles. For simplicity, the particles are considered to be spheres of radius $r_\varepsilon << \varepsilon$, such that:

$$\lim_{\varepsilon \to 0} \gamma_\varepsilon = \gamma \in]0, +\infty[, \tag{1}$$

where $\gamma_\varepsilon := r_\varepsilon / \varepsilon^3$, which corresponds to the well-known critical case of vanishing fine substructures homogenization [10].

We balance this assumption, which obviously means that the suspension has vanishing volume, by supposing that the total mass of the suspension is of unity order. This simplified structure permits the accurate establishment of the macroscopic equations by the control-zone method [6, 7] of the homogenization theory for fine-scale substructures.

Diffusion occurs naturally in many industrial and geophysical problems, particularly in oil recovery, earth pollution, phase transition, chemical and nuclear processes. The study of diffusion in micro-periodic binary structures has a crucial point in the interaction between the microscopic and macroscopic levels and particularly in the way the former influences the latter. Always an appropriate choice of the relative scales is needed. To give a flavor of what may be considered, we refer to the pioneering work [10] where the appearance of an extra term in the limit procedure is responsible for a change in the nature of the mathematical problem and is linked to a critical size of the inclusions. Later [9] showed how this could be generalized to the N-dimensional case for non linear operators satisfying classical properties of polynomial growth and coercivity. Since then, the notion of non local effects has been developed in a way that is closer to the present point of view in [4], [8], [6] and [7].

Since the fundamental work [10], an important step in the homogenization of vanishing fine substructures was accomplished by [3]. A slightly different approach [8] uses Dirichlet forms involving non classical measures in the spirit of [15]. However, the main drawback of this method lies in its essential use of the Maximum Principle, which was avoided in [4] for elastic fibers, and later in [6] where the case of spherical symmetry is solved. The dependence on the geometrical symmetry was overcome in [7]. The asymptotic behavior of highly heterogeneous media has also been considered in the framework of homogenization when the coefficient of one component is vanishing and both components have volumes of unity order: see the derivation of a double porosity model for a single phase flow by [2] and the application of two-scale convergence in order to model diffusion processes in [1].

The paper is organized as follows. Section 2 is devoted to the main notations and to the description of the functional framework (15)–(19). In Section 3, we introduce the specific hypotheses on the control zone in relation with the assumption of finite diffusivities. The homogenization is performed in Section 4: while equation (58) easily follows from the variational formulation, the deduction of equation (67) is our main contribution, where the method reveals its potential.

2 The diffusion problem

We consider $\Omega \subseteq \mathbf{R}^3$ a bounded Lipschitz domain occupied by a mixture of two different materials, one of them forming the ambiental connected phase and the other being concentrated in a periodical suspension of small spherical particles. Let us denote

$$Y := \left(-\frac{1}{2}, +\frac{1}{2}\right)^3. \tag{2}$$

$$Y_\varepsilon^k := \varepsilon k + \varepsilon Y, \quad k \in \mathbf{Z}^3. \tag{3}$$

$$\mathbf{Z}_\varepsilon := \{k \in \mathbf{Z}^3, \quad Y_\varepsilon^k \subset \Omega\}, \quad \Omega_{Y_\varepsilon} := \cup_{k \in \mathbf{Z}_\varepsilon} Y_\varepsilon^k. \tag{4}$$

The suspension is defined by the following reunion

$$D_\varepsilon := \cup_{k \in \mathbf{Z}_\varepsilon} B_{r_\varepsilon}^k, \quad B_{r_\varepsilon}^k = B(\varepsilon k, r_\varepsilon), \quad k \in \mathbf{Z}_\varepsilon, \tag{5}$$

where $0 < r_\varepsilon << \varepsilon$.

The fluid domain is given by

$$\Omega_\varepsilon = \Omega \setminus D_\varepsilon. \tag{6}$$

We also use the following notation for the cylindrical time-domain:

$$\Omega^T := \Omega \times]0, T[; \tag{7}$$

similar definitions for Ω_ε^T, $\Omega_{Y_\varepsilon}^T$ and D_ε^T.

We consider the problem which governs the diffusion process throughout a binary mixture, where we consider that the density of the spherical particles is much higher than that of the surrounding phase, such that the volume of the suspension is vanishing while its mass is of unity order. This can be described by taking the relative mass density of the form:

$$\rho^\varepsilon(x) = \begin{cases} a/|D_\varepsilon| & \text{if} \quad x \in D_\varepsilon \\ 1 & \text{if} \quad x \in \Omega_\varepsilon \end{cases} \tag{8}$$

where $a > 0$.

Denoting by $b > 0$ the relative diffusivity of the suspension, then, assuming without loss of generality that $|\Omega| = 1$, the non-dimensional form of the governing system is the following:

$$\rho^\varepsilon \frac{\partial u^\varepsilon}{\partial t} - \operatorname{div}(k^\varepsilon \nabla u^\varepsilon) = \rho^\varepsilon f^\varepsilon \quad \text{in} \quad \Omega^T \tag{9}$$

$$[u^\varepsilon]_\varepsilon = 0 \quad \text{on} \quad \partial D_\varepsilon^T \tag{10}$$

$$[k^\varepsilon \nabla u^\varepsilon]_\varepsilon n = 0 \quad \text{on} \quad \partial D_\varepsilon^T \tag{11}$$

$$u^\varepsilon = 0 \quad \text{on} \quad \partial \Omega^T \tag{12}$$

$$u^\varepsilon(0) = u_0^\varepsilon \quad \text{in} \quad \Omega \tag{13}$$

where $[\cdot]_\varepsilon$ is the jump across the interface ∂D_ε, n is the normal on ∂D_ε in the outward direction, $f^\varepsilon \in L^2(\Omega^T)$, $u_0^\varepsilon \in H_0^1(\Omega)$ and

$$k^\varepsilon(x) = \begin{cases} b & \text{if} \quad x \in D_\varepsilon, \\ 1 & \text{if} \quad x \in \Omega_\varepsilon. \end{cases} \tag{14}$$

Let H_ε be the Hilbert space $L^2(\Omega)$ endowed with the scalar product

$$(u, v)_{H_\varepsilon} := (\rho^\varepsilon u, v)_\Omega \tag{15}$$

As $H_0^1(\Omega)$ is dense in H_ε for any fixed $\varepsilon > 0$, we can set

$$H_0^1(\Omega) \subseteq H_\varepsilon \simeq H_\varepsilon' \subseteq H^{-1}(\Omega) \tag{16}$$

with continuous embeddings.

As the form

$$k_\varepsilon(u, v) = (k^\varepsilon \nabla u, \nabla v)_\Omega, \quad u, v \in H_0^1(\Omega) \tag{17}$$

is symmetric, bounded and coercive on $H_0^1(\Omega)$, the weak formulation of the problem (9)-(13) is the following:

Find $u^\varepsilon \in L^2(0,T;H_0^1(\Omega))$ such that

$$\frac{\partial u^\varepsilon}{\partial t} + K_\varepsilon u^\varepsilon = f^\varepsilon \quad \text{in} \quad L^2(\Omega^T), \tag{18}$$

$$u^\varepsilon(0) = u_0^\varepsilon \quad \text{in} \quad C^0([0,T];H_\varepsilon), \tag{19}$$

where K_ε is the operator associated with the form k_ε by the first representation theorem in H_ε.

Concerning this problem, we have a classical result of regularity (see [11], Chap. XV), which insures the existence and uniqueness of the solution of problem (18)–(19).

3 Tools of the control-zone method

The set of control-sequences is defined by

$$\mathcal{R} = \{(R_\varepsilon)_{\varepsilon>0}, \quad r_\varepsilon << R_\varepsilon << \varepsilon\}$$

that is $(R_\varepsilon)_{\varepsilon>0} \in \mathcal{R}$ iff

$$\lim_{\varepsilon\to 0} \frac{r_\varepsilon}{R_\varepsilon} = \lim_{\varepsilon\to 0} \frac{R_\varepsilon}{\varepsilon} = 0. \tag{20}$$

We have to remark that $\mathcal{R}$ is an infinite set, this property being insured by the assumption (1).

For any $(R_\varepsilon)_{\varepsilon>0} \in \mathcal{R}$, we define the control-zone of the suspension by:

$$D_{R_\varepsilon} := \cup_{k\in\mathbf{Z}_\varepsilon} B^k_{R_\varepsilon}, \quad \text{where} \quad B^k_{R_\varepsilon} := B(\varepsilon k, R_\varepsilon).$$

The specific operator of the method, $G_{R_\varepsilon} : L^2(0,T;H_0^1(\Omega)) \to L^2(\Omega^T)$, is defined by

$$G_{R_\varepsilon}(\theta)(x,t) = \sum_{k\in\mathbf{Z}_\varepsilon} \left(⨍_{\partial B^k_{R_\varepsilon}} \theta(y,t)\, d\sigma_y \right) 1_{Y^k_\varepsilon}(x). \tag{21}$$

We remind here two properties, already proved in [5]:

Proposition 3.1 *If $(R_\varepsilon)_{\varepsilon>0} \in \mathcal{R}$, then for every $\theta \in L^2(0,T;H_0^1(\Omega))$ we have*

$$|\theta - G_{R_\varepsilon}(\theta)|_{L^2(\Omega^T_{Y_\varepsilon})} \le C \left(\frac{\varepsilon^3}{R_\varepsilon}\right)^{1/2} |\nabla\theta|_{L^2(\Omega^T)}. \tag{22}$$

Moreover:

$$|G_{R_\varepsilon}(\theta)|^2_{L^2(\Omega^T)} = \int_0^T ⨍_{D_\varepsilon} |G_{R_\varepsilon}(\theta)|^2. \tag{23}$$

For any $(R_\varepsilon)_{\varepsilon>0} \in \mathcal{R}$, we define $w^\varepsilon \in L^2(0,T;H_0^1(\Omega))$, the key test-function associated to our control-zone, by

$$w^\varepsilon(t,x) := \begin{cases} W^\varepsilon(t,|x-\varepsilon k|), & (t,x) \in (B_{R_\varepsilon}^k)^T, \quad k \in \mathbf{Z}_\varepsilon, \\ 0, & (t,x) \in (\Omega \setminus D_{R_\varepsilon})^T \end{cases} \tag{24}$$

where, denoting by $B_{R_\varepsilon} = B(0,R_\varepsilon)$, $B_{r_\varepsilon} = B(0,r_\varepsilon)$, $C_\varepsilon = (1 - r_\varepsilon/R_\varepsilon)^{-1}$ and

$$W_0^\varepsilon(y) = \begin{cases} C_\varepsilon \left(\dfrac{r_\varepsilon}{|y|} - \dfrac{r_\varepsilon}{R_\varepsilon} \right), & y \in B_{R_\varepsilon} \setminus B_{r_\varepsilon}, \\ 1, & y \in B_{r_\varepsilon}, \end{cases}$$

we have $W^\varepsilon \in L^2(0,T;H_0^1(B_{R_\varepsilon}))$ as the solution of the system:

$$\rho^\varepsilon \frac{\partial W^\varepsilon}{\partial t} - \operatorname{div}(k_\varepsilon \nabla W^\varepsilon) = 0 \quad \text{in} \quad B_{R_\varepsilon}^T, \tag{25}$$

$$[W^\varepsilon] = [k_\varepsilon \nabla W^\varepsilon] \cdot n = 0 \quad \text{on} \quad \partial B_{r_\varepsilon}^T, \tag{26}$$

$$W^\varepsilon = 0 \quad \text{on} \quad \partial B_{R_\varepsilon}^T, \tag{27}$$

$$W^\varepsilon(0) = W_0^\varepsilon \quad \text{in} \quad B_{R_\varepsilon}. \tag{28}$$

Remark 3.2 *Denoting by $\star$ the convolution with respect to the time variable and by*

$$\alpha := \sqrt{\frac{3a}{4\pi\gamma b}}, \tag{29}$$

straightforward computations yield:

$$W^\varepsilon(t,y) = W_0^\varepsilon(y) + \int_0^t (h_\varepsilon \star S_\varepsilon(y))(s)ds \tag{30}$$

where, for $p \in \mathrm{C}$, $\mathrm{Re}(p) > 0$, h_ε and S_ε have the following Laplace transforms:

$$\hat{h}_\varepsilon(p) = \frac{C_\varepsilon}{(b-1) - b\alpha\sqrt{p}\coth(\alpha\sqrt{p}) - C_\varepsilon \dfrac{r_\varepsilon}{R_\varepsilon} \dfrac{(R_\varepsilon - r_\varepsilon)\sqrt{p}}{\tanh((R_\varepsilon - r_\varepsilon)\sqrt{p})}},$$

$$\hat{S}_\varepsilon(p,y) = \begin{cases} \dfrac{\alpha\sqrt{p}}{\sinh(\alpha\sqrt{p})}, & \textit{for} \quad y = 0, \\[2ex] \dfrac{r_\varepsilon}{|y|} \dfrac{\sinh(|y|\alpha\sqrt{p}/r_\varepsilon)}{\sinh(\alpha\sqrt{p})}, & \textit{for} \quad y \in B_{r_\varepsilon} \setminus \{0\}, \\[2ex] \dfrac{r_\varepsilon}{|y|} \dfrac{\sinh((R_\varepsilon - |y|)\sqrt{p})}{\sinh((R_\varepsilon - r_\varepsilon)\sqrt{p})}, & \textit{for} \quad y \in B_{r_\varepsilon} \setminus B_{r_\varepsilon}. \end{cases}$$

Moreover, we obtain

$$\nabla W^\varepsilon(t,y)n = -C_\varepsilon \frac{r_\varepsilon}{{R_\varepsilon}^2} F_\varepsilon(t), \quad \textit{for} \quad y \in \partial B_{R_\varepsilon}, \tag{31}$$

where $F_\varepsilon \in L^\infty(0,T)$ is defined after its Laplace transform:

$$p\hat{F}_\varepsilon(p) = 1 + \hat{h}_\varepsilon(p)\,\frac{(R_\varepsilon - r_\varepsilon)\sqrt{p}}{\tanh((R_\varepsilon - r_\varepsilon)\sqrt{p})}.$$

Proposition 3.3 *For any $(R_\varepsilon)_{\varepsilon>0} \in \mathcal{R}$, we have*

$$(w^\varepsilon)_{\varepsilon>0} \quad \textit{is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)), \tag{32}$$

$$w^\varepsilon \to 0 \quad \textit{in} \quad L^\infty(0,T;L^2(\Omega)). \tag{33}$$

Next, we remind the properties of the main operators used in the homogenization of vanishing fine substructures.

Definition 3.4 *Let m_ε and M_ε be defined for every $\varphi \in L^2(\Omega)$ by:*

$$m_\varepsilon(\varphi)(x) := \sum_{k\in\mathbf{Z}_\varepsilon} m_\varepsilon^k(\varphi)1_{Y_\varepsilon^k}, \quad m_\varepsilon^k(\varphi) = \fint_{B_{r_\varepsilon}^k} \varphi,$$

$$M_\varepsilon(\varphi)(x) := \sum_{k\in\mathbf{Z}_\varepsilon} M_\varepsilon^k(\varphi)1_{B_{r_\varepsilon}^k}, \quad M_\varepsilon^k(\varphi) = \fint_{Y_\varepsilon^k} \varphi.$$

This definition obviously holds for every $\varphi \in C_c(\Omega)$. It extends to $\varphi \in L^2(\Omega)$ thanks to a density argument and to the following estimates:

$$|m_\varepsilon(\varphi)|_\Omega^2 \le \fint_{D_\varepsilon} \varphi^2 \tag{34}$$

$$\fint_{D_\varepsilon} M_\varepsilon(\varphi)^2 \le \frac{1}{|\Omega_{Y_\varepsilon}|}|\varphi|_{\Omega_{Y_\varepsilon}}^2.$$

Moreover, both operators are linked through the following duality relation:

$$\forall \varphi,\, \psi \in L^2(\Omega), \quad \int_\Omega m_\varepsilon(\varphi)\psi = |\Omega_{Y_\varepsilon}| \fint_{D_\varepsilon} M_\varepsilon(\psi). \tag{35}$$

Remark 3.5 *Using the Mean Value Theorem, we easily find that, for every $\psi \in C_c(\Omega)$,*

$$\lim_{\varepsilon\to 0} |m_\varepsilon(\psi) - \psi|_{L^\infty(\Omega)} = 0, \quad \lim_{\varepsilon\to 0} |M_\varepsilon(\psi) - \psi|_{L^\infty(D_\varepsilon)} = 0. \tag{36}$$

Lemma 3.6 *For any $\varphi \in L^2(0,T;C_c(\Omega))$, we have:*

$$\lim_{\varepsilon\to 0} \int_0^T \fint_{D_\varepsilon} |\varphi - M_\varepsilon(\varphi)|^2 dx dt = 0.$$

Proof. Notice that

$$⨍_{D_\varepsilon} |\varphi - M_\varepsilon(\varphi)|^2 dx = \frac{1}{|D_\varepsilon|} \sum_{k \in \mathbf{Z}_\varepsilon} \int_{B(\varepsilon k, r_\varepsilon)} |\varphi - ⨍_{Y_\varepsilon^k} \varphi \, dy|^2 \, dx.$$

As $\text{card}(\mathbf{Z}_\varepsilon) \simeq \dfrac{|\Omega|}{\varepsilon^3}$, then $|B(0,r_\varepsilon)| \dfrac{\text{card}(\mathbf{Z}_\varepsilon)}{|D_\varepsilon|} \to |\Omega| = 1$ and by the uniform continuity of φ on Ω it follows the convergence to 0 a.e. on $[0,T]$. Lebesgue's dominated convergence theorem achieves the result. ■

Proposition 3.7 *There exists $C > 0$, independent of ε, such that for any $\theta \in L^2(0,T;H_0^1(\Omega))$ there holds true:*

$$\int_0^T ⨍_{D_\varepsilon} |\theta|^2 \, dxdt \leq C |\nabla\theta|^2_{L^2(\Omega^T)}.$$

Our procedure apels to the following property, for which we present a simple proof; it can also be deduced from Lemma A2 of [3].

Theorem 3.8 *Assume (1) and let $u_\varepsilon \in L^2(\Omega)$ satisfy the following uniform estimate:*

$$⨍_{D_\varepsilon} |u_\varepsilon|^2 \leq C, \quad \forall \varepsilon > 0. \tag{37}$$

Then, there exists $v \in L^2(\Omega)$ such that:

$$⨍_{D_\varepsilon} u_\varepsilon \varphi \to \int_\Omega v\varphi, \quad \forall \varphi \in H_0^1(\Omega), \tag{38}$$

on some subsequence.

Proof. The hypothesis (37) implies that the sequence $\{m_\varepsilon(u_\varepsilon)\}_\varepsilon$ is bounded in $L^2(\Omega)$ thanks to (34). Thus, there exists $v \in L^2(\Omega)$ such that

$$m_\varepsilon(u_\varepsilon) \rightharpoonup v \quad \text{weakly in} \quad L^2(\Omega). \tag{39}$$

Let $\varphi \in C_c(\Omega)$. Then, using (35), we get

$$⨍_{D_\varepsilon} u_\varepsilon \varphi = ⨍_{D_\varepsilon} u_\varepsilon(\varphi - M_\varepsilon(\varphi)) + ⨍_{D_\varepsilon} u_\varepsilon M_\varepsilon(\varphi) = I_\varepsilon + \frac{1}{|\Omega_{Y_\varepsilon}|} \int_\Omega m_\varepsilon(u_\varepsilon)\varphi$$

where

$$|I_\varepsilon| = \left| ⨍_{D_\varepsilon} u_\varepsilon(\varphi - M_\varepsilon(\varphi)) \right| \leq C \left(⨍_{D_\varepsilon} |\varphi - M_\varepsilon(\varphi)|^2 \right)^{1/2} \leq C|\varphi - M_\varepsilon(\varphi)|_{L^\infty(D_\varepsilon)}.$$

The second part in (36) yields that $I_\varepsilon \to 0$ and thus we have proved (38) for any $\varphi \in C_c(\Omega)$. As $C_c(\Omega)$ is dense in $H_0^1(\Omega)$, the proof is completed by Proposition 3.7. ■

4 The homogenization procedure

In the following, we present the hypotheses under which we study the asymptotical behaviour of u^ε (as $\varepsilon \to 0$).

First, we assume that there exist $f \in L^2(\Omega^T)$ and $u_0 \in L^2(\Omega)$ such that

$$\rho^\varepsilon f^\varepsilon \rightharpoonup f \quad \text{in} \quad L^2(\Omega^T), \tag{40}$$

$$u_0^\varepsilon \rightharpoonup u_0 \quad \text{in} \quad L^2(\Omega), \tag{41}$$

and that there exist $C > 0$ (independent of ε) and $v_0 \in L^2(\Omega)$ such that

$$⨍_{D_\varepsilon} |u_0^\varepsilon|^2 dx \le C \tag{42}$$

$$\frac{1}{|D_\varepsilon|} u_0^\varepsilon \chi_{D_\varepsilon} \rightharpoonup v_0 \quad \text{in} \quad \mathcal{D}'(\Omega) \tag{43}$$

where, for any $D \subset \Omega$, we denote

$$⨍_D \cdot\, dx = \frac{1}{|D|} \int_D \cdot dx.$$

Remark 4.1 *As u_0^ε satisfies (42) then, from (38) it follows that (43) holds at least on some subsequence.*

Using only these assumptions, we readily obtain:

Proposition 4.2 *If u^ε is the solution of the problem (18)–(19), then from (40)–(43) it follows:*

$$(u^\varepsilon)_{\varepsilon>0} \quad \textit{is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)). \tag{44}$$

$$\exists C > 0 \quad \textit{such that} \quad ⨍_{D_\varepsilon} |u^\varepsilon(t)|^2 \le C, \quad \forall \varepsilon > 0, \quad \textit{for a.a. } t \in [0,T]. \tag{45}$$

Proof. Multiplying equation (18) by u^ε and integrating over Ω^t for any $t \in]0,T[$, we get:

$$\frac{1}{2}\left(|u^\varepsilon(t)|^2_{\Omega_\varepsilon} + a ⨍_{D_\varepsilon} |u^\varepsilon(t)|^2 \right) + b \int_0^t |\nabla u^\varepsilon(t)|^2_{D_\varepsilon} ds + \int_0^t |\nabla u^\varepsilon(t)|^2_{\Omega_\varepsilon} ds =$$

$$= \int_0^t \int_\Omega \rho^\varepsilon f^\varepsilon(s) u^\varepsilon(s) ds + \frac{1}{2}\left(|u_0^\varepsilon|^2_{\Omega_\varepsilon} + a ⨍_{D_\varepsilon} |u_0^\varepsilon|^2 \right).$$

Notice that (42) yields:

$$|u_0^\varepsilon|^2_\Omega + a ⨍_{D_\varepsilon} |u_0^\varepsilon|^2 dx \le C.$$

Using the Poincaré-Friedrichs inequality in Ω and (40), we have:

$$\int_0^t \int_\Omega \rho^\varepsilon f^\varepsilon(s) u^\varepsilon(s) ds \le C \int_0^t |\rho^\varepsilon f^\varepsilon|_\Omega |\nabla u^\varepsilon|_\Omega ds \le C |\nabla u^\varepsilon|_{\Omega^t}.$$

There results:

$$|u^\varepsilon(t)|^2_{\Omega_\varepsilon} + a \fint_{D_\varepsilon} |u^\varepsilon(t)|^2 + b\int_0^t |\nabla u^\varepsilon|^2_{D_\varepsilon} ds + \int_0^t |\nabla u^\varepsilon|^2_{\Omega_\varepsilon} ds \le C$$

and the proof is completed. ■

In order to prove the convergence of the homogenization process, we have to add the hypotheses which describe the behaviour of the data versus the key test-function associated to the control-zone:

There exist $(R_\varepsilon)_{\varepsilon>0} \in \mathcal{R}$, $w_0 \in C^0([0,T]; L^2(\Omega))$ and $g \in L^2(\Omega^T)$ for which:

$$\int_\Omega \rho^\varepsilon w_\varepsilon(t) u_0^\varepsilon \varphi \to \int_\Omega w_0(t)\varphi, \quad \text{in} \quad C^0([0,T]), \quad \forall \varphi \in \mathcal{D}(\Omega), \tag{46}$$

$$\int_\Omega \rho^\varepsilon (f^\varepsilon \star w_\varepsilon)(t)\varphi \to \int_\Omega \int_0^t g(s)\varphi ds \quad \text{in} \quad \mathcal{D}'(0,T), \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{47}$$

A preliminary result is the following:

Proposition 4.3 *If u^ε is the solution of the problem (18)–(19), then there exist $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ and $v \in L^2(\Omega^T)$ such that:*

$$u^\varepsilon \overset{\star}{\rightharpoonup} u \quad in \quad L^\infty(0,T;L^2(\Omega)) \tag{48}$$

$$u^\varepsilon \rightharpoonup u \quad in \quad L^2(0,T;H_0^1(\Omega)) \tag{49}$$

$$G_{R_\varepsilon}(u^\varepsilon) \to u \quad in \quad L^2(\Omega^T) \tag{50}$$

$$m_\varepsilon(u^\varepsilon) \rightharpoonup v \quad in \quad L^2(\Omega^T) \tag{51}$$

on some subsequence.

Proof. From (44), we get, on some subsequence, the convergences (48) and (49). Moreover, we have:

$$|u - G_{R_\varepsilon}(u^\varepsilon)|^2_{\Omega^T} = |u|^2_{\Omega^T \setminus \Omega^T_{Y_\varepsilon}} + |u - G_{R_\varepsilon}(u^\varepsilon)|^2_{\Omega^T_{Y_\varepsilon}} \tag{52}$$

where:

$$\begin{aligned} |u - G_{R_\varepsilon}(u^\varepsilon)|_{\Omega^T_{Y_\varepsilon}} &\le |u - u^\varepsilon|_{\Omega^T_{Y_\varepsilon}} + |u^\varepsilon - G_{R_\varepsilon}(u^\varepsilon)|_{\Omega^T_{Y_\varepsilon}} \\ &\le |u - u^\varepsilon|_{\Omega^T} + |u^\varepsilon - G_{R_\varepsilon}(u^\varepsilon)|_{\Omega^T_{Y_\varepsilon}} \end{aligned} \tag{53}$$

and (22) yields:

$$|u^\varepsilon - G_{R_\varepsilon}(u^\varepsilon)|^2_{\Omega^T_{Y_\varepsilon}} \le C\frac{\varepsilon^3}{R_\varepsilon}|\nabla u^\varepsilon|^2_{\Omega^T} = C\frac{\varepsilon^3}{r_\varepsilon}\frac{r_\varepsilon}{R_\varepsilon}|\nabla u^\varepsilon|^2_{\Omega^T} \le C\frac{r_\varepsilon}{R_\varepsilon}$$

and thus:

$$\lim_{\varepsilon\to 0} |u^\varepsilon - G_{R_\varepsilon}(u^\varepsilon)|^2_{\Omega^T_{Y_\varepsilon}} = 0.$$

As (49) implies that

$$u^\varepsilon \to u \quad \text{in} \quad L^2(\Omega^T) \tag{54}$$

the right-hand side of (53) tends to zero as $\varepsilon \to 0$, that is:

$$\lim_{\varepsilon\to 0} |u - G_{R_\varepsilon}(u^\varepsilon)|_{\Omega^T_{Y_\varepsilon}} = 0.$$

After substitution into the right-hand side of (52), and taking into account that

$$\lim_{\varepsilon\to 0} |\Omega^T \setminus \Omega^T_{Y_\varepsilon}| = 0,$$

we obtain (50), that is,

$$G_{R_\varepsilon}(u^\varepsilon) \to u \quad \text{in} \quad L^2(\Omega^T). \tag{55}$$

In order to prove (51), we apply (39) of Theorem 3.8, and the proof is completed. ■

Remark 4.4 *As a consequence of (51) and Theorem 3.8, we obviously have:*

$$\int_0^T \fint_{D_\varepsilon} u^\varepsilon \varphi dxdt \to \int_{\Omega^T} v\varphi dxdt, \quad \forall \varphi \in L^2(0,T;H_0^1(\Omega)), \tag{56}$$

on the subsequence mentioned by Proposition 4.3. Taking into account (46) and (43), Theorem 3.8 implies also:

$$w_0(0) = av_0. \tag{57}$$

We are in the position to state our first result:

Theorem 4.5 *The limits $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ and $v \in L^2(\Omega^T)$ of (48)–(51) verify (in a weak sense) the following problem:*

$$\begin{aligned} a\frac{\partial v}{\partial t} + \frac{\partial u}{\partial t} - \Delta u &= f \quad in \quad \Omega^T, \qquad (58)\\ av(0) + u(0) &= av_0 + u_0 \quad in \quad \Omega \qquad (59) \end{aligned}$$

Moreover, there holds $u \in C^0([0,T];L^2(\Omega))$ and $v \in C^0([0,T];H^{-1}(\Omega))$; this is the sense of (59).

Proof. Multiplying (18) by $\psi\eta$, where $\psi \in \mathcal{D}(\Omega)$ and $\eta \in \mathcal{D}([0,T[)$, and integrating it over Ω^T, we obtain:

$$-\int_{\Omega^T} \rho^\varepsilon u^\varepsilon \psi\eta' + \int_{\Omega^T} k^\varepsilon \nabla u^\varepsilon (\nabla\psi)\eta = \int_{\Omega^T} \rho^\varepsilon f^\varepsilon \psi\eta + \int_\Omega \rho^\varepsilon u_0^\varepsilon \psi\eta(0). \tag{60}$$

First, let us notice that

$$\int_{\Omega^T} \rho^\varepsilon u^\varepsilon \psi\eta' = \int_0^T \int_\Omega \chi_{\Omega_\varepsilon} u^\varepsilon \psi(x)\eta' + a\int_0^T \fint_{D_\varepsilon} u^\varepsilon \psi\eta'.$$

Using Proposition 4.4, we easily get

$$\lim_{\varepsilon\to 0} \int_{\Omega^T} \rho^\varepsilon u^\varepsilon \psi\eta' = \int_{\Omega^T} u\varphi\eta' + a\int_{\Omega^T} v\psi\eta'.$$

For the second term of (60), we have

$$\int_{\Omega^T} k_\varepsilon \nabla u^\varepsilon \nabla \psi \eta' = b \int_{D_\varepsilon} \nabla u^\varepsilon \nabla \psi \eta' + \int_{\Omega_\varepsilon} \nabla u^\varepsilon \nabla \psi \eta'$$

where

$$\int_{\Omega_\varepsilon^T} \nabla u^\varepsilon \nabla \psi \eta' = \int_{\Omega^T} 1_{\Omega_\varepsilon} \nabla u^\varepsilon \nabla \psi \eta' \to \int_{\Omega^T} \nabla u \nabla \psi \eta',$$

and

$$\left| b \int_{D_\varepsilon^T} \nabla u^\varepsilon \nabla \psi \eta' \right| \leq |\nabla u^\varepsilon|_\Omega |\nabla \psi|_{D_\varepsilon} |\eta'|_\infty \leq C|D_\varepsilon| \to 0,$$

which proves that

$$\lim_{\varepsilon \to 0} \int_{\Omega^T} k_\varepsilon \nabla u^\varepsilon \nabla \psi \eta' = \int_{\Omega^T} \nabla u \nabla \psi \eta'.$$

As for the right-hand side, using hypothesis (40), we have:

$$\lim_{\varepsilon \to 0} \int_{\Omega^T} \rho^\varepsilon f^\varepsilon \psi \eta = \int_{\Omega^T} f \psi \eta.$$

Finally, noticing that

$$\int_\Omega \rho^\varepsilon u_0^\varepsilon \psi \eta(0) = \int_{\Omega_\varepsilon} u_0^\varepsilon \psi \eta(0) + a \fint_{D_\varepsilon} u_0^\varepsilon \psi \eta(0),$$

and using the hypotheses (41) and (43), we pass to the limit with the same arguments as above, obtaining:

$$\lim_{\varepsilon \to 0} \int_\Omega \rho^\varepsilon u_0^\varepsilon \psi \eta(0) = \eta(0) \int_\Omega (u_0 + a v_0) \psi,$$

which achieves the proof. ■

Proposition 4.6 *If u^ε is the solution of the problem (18)–(19), then we have on the subsequence of Proposition 4.3:*

$$\begin{aligned} &\left\{ \int_\Omega \rho^\varepsilon \frac{\partial u^\varepsilon}{\partial t} \star w_\varepsilon m_\varepsilon(\varphi) - \int_\Omega u^\varepsilon \star \left(\rho^\varepsilon \frac{\partial w_\varepsilon}{\partial t} \right) m_\varepsilon(\varphi) \right\} \rightharpoonup \\ &\rightharpoonup a \int_\Omega \eta v \varphi - \int_\Omega w_0 \varphi \quad \textit{weakly in} \quad L^2(0,T), \quad \forall \varphi \in \mathcal{D}(\Omega). \end{aligned} \tag{61}$$

Proof. Let $\varphi \in \mathcal{D}(\Omega)$; we have

$$\int_\Omega \rho^\varepsilon \frac{\partial u^\varepsilon}{\partial t} \star w_\varepsilon m_\varepsilon(\varphi) = \int_\Omega \rho^\varepsilon \left(u^\varepsilon \star \frac{\partial w_\varepsilon}{\partial t} + w_{R_\varepsilon} u^\varepsilon - w_\varepsilon u_0^\varepsilon \right) m_\varepsilon(\varphi) =$$

$$= \int_\Omega u^\varepsilon \star \left(\rho^\varepsilon \frac{\partial w_\varepsilon}{\partial t} \right) m_\varepsilon(\varphi) + a \fint_{D_\varepsilon} u^\varepsilon m_\varepsilon(\varphi) + \int_{D_{R_\varepsilon} \setminus D_\varepsilon} w_{R_\varepsilon} u^\varepsilon m_\varepsilon(\varphi) - \int_\Omega \rho^\varepsilon w_\varepsilon u_0^\varepsilon m_\varepsilon(\varphi).$$

As

$$\left| \int_{D_{R_\varepsilon} \setminus D_\varepsilon} w_{R_\varepsilon} u^\varepsilon m_\varepsilon(\varphi) \right| \leq |u^\varepsilon|_{L^\infty(0,T;L^2(\Omega))} |m_\varepsilon(\varphi)|_{D_{R_\varepsilon} \setminus D_\varepsilon} \leq C|D_{R_\varepsilon}| \to 0,$$

we obtain

$$\lim_{\varepsilon\to 0}\int_{D_{R_\varepsilon}\setminus D_\varepsilon} w_{R_\varepsilon} u^\varepsilon m_\varepsilon(\varphi) = 0 \quad \text{in} \quad L^2(0,T).$$

The following estimate

$$\left|\int_{D_\varepsilon} w_\varepsilon u_0^\varepsilon (m_\varepsilon(\varphi)-\varphi)\right| \le \left(⨍_{D_\varepsilon} |w_\varepsilon|^2\right)^{1/2} \left(⨍_{D_\varepsilon} |u_0^\varepsilon|^2\right)^{1/2} |m_\varepsilon(\varphi)-\varphi)|_{L^\infty(D_\varepsilon)}$$
$$\le C|m_\varepsilon(\varphi)-\varphi)|_{L^\infty(D_\varepsilon)}$$

and Remark 3.5 yield also

$$\lim_{\varepsilon\to 0}\int_\Omega \rho^\varepsilon w_\varepsilon u_0^\varepsilon \varphi = \int_\Omega w_0\varphi \quad \text{in} \quad L^2(0,T),$$

and the proof is completed.

■

Proposition 4.7 *If u^ε is the solution of the problem (18)–(19), then, still on the subsequence of Proposition 4.3, we have for $\forall\varphi \in \mathcal{D}(\Omega)$:*

$$\int_\Omega \Bigg\{ -\mathrm{div}(k_\varepsilon \nabla u^\varepsilon) \star w_\varepsilon m_\varepsilon(\varphi) - ⨍_{D_\varepsilon} u^\varepsilon \star (-b\Delta w_\varepsilon) m_\varepsilon(\varphi)$$
$$- \int_{D_{R_\varepsilon}\setminus D_\varepsilon} u^\varepsilon \star (-\Delta w_\varepsilon) m_\varepsilon(\varphi) \Bigg\} \rightharpoonup -4\pi\gamma \int_\Omega F_{\alpha,b} \star u\, \varphi \quad \textit{weakly in} \quad L^2(0,T), \tag{62}$$

where

$$F_{\alpha,b}(t) = 1 + \int_0^t h_{\alpha,b}(s)ds \tag{63}$$

and $h_{\alpha,b}$ is defined after its Laplace transform:

$$\hat{h}_{\alpha,b}(p) = \frac{1}{(b-1) - b\alpha\sqrt{p}\coth(\alpha\sqrt{p})}. \tag{64}$$

Remark 4.8 *$h_{\alpha,b}$ can be expressed in terms of the Theta functions of Jacobi (see [12],[13]). For instance, in the case $b=1$, we find straightly that*

$$h_{\alpha,1}(t) = \frac{1}{\alpha\sqrt{\pi t}} \sum_{n=-\infty}^{+\infty} (-1)^n \exp(-n^2\alpha^2/t).$$

Proof of Proposition 4.7. For $\varphi \in \mathcal{D}(\Omega)$, we have:

$$\int_\Omega \{-\mathrm{div}(k_\varepsilon\nabla u^\varepsilon) \star w_\varepsilon m_\varepsilon(\varphi)\} = \int_0^t \int_\Omega k_\varepsilon \nabla u^\varepsilon(t-s)\nabla w_\varepsilon(s) m_\varepsilon(\varphi)$$

with

$$b\int_0^t \int_{D_\varepsilon} \nabla u^\varepsilon(t-s)\nabla w_\varepsilon(s) m_\varepsilon(\varphi) = \int_0^t \int_{D_\varepsilon} u^\varepsilon(t-s)(-b\Delta w_\varepsilon(s)) m_\varepsilon(\varphi) +$$

$$+\sum_{k\in\mathbf{Z}_\varepsilon}\int_0^t\int_{\partial B^k_{r_\varepsilon}} u^\varepsilon(t-s)(b\frac{\partial w_\varepsilon}{\partial n}(s))m^k_\varepsilon(\varphi),$$

$$\int_0^t\int_{\Omega_\varepsilon}\nabla u^\varepsilon(t-s)\nabla w_\varepsilon(s)m_\varepsilon(\varphi)=\int_0^t\int_{D_{R_\varepsilon}\setminus D_\varepsilon} u^\varepsilon(t-s)(-\Delta w_\varepsilon(s))m_\varepsilon(\varphi)+$$

$$+\sum_{k\in\mathbf{Z}_\varepsilon}\int_0^t\int_{\partial B^k_{R_\varepsilon}} u^\varepsilon(t-s)(\frac{\partial w_\varepsilon}{\partial n}(s))m^k_\varepsilon(\varphi)-\sum_{k\in\mathbf{Z}_\varepsilon}\int_0^t\int_{\partial B^k_{r_\varepsilon}} u^\varepsilon(t-s)(\frac{\partial w_\varepsilon}{\partial n}(s))m^k_\varepsilon(\varphi).$$

Notice that (31) yields

$$\frac{\partial w_\varepsilon}{\partial n}(s,x)=\frac{\partial W_\varepsilon}{\partial n}(s,x-\varepsilon k)=-C_\varepsilon\frac{r_\varepsilon}{{R_\varepsilon}^2}F_\varepsilon(s)\quad\text{on}\quad\partial B^k_{R_\varepsilon},$$

and thus, there holds in $L^2(0,T)$:

$$\begin{aligned}\int_\Omega\Big\{&-\mathrm{div}(k_\varepsilon\nabla u^\varepsilon)\star w_\varepsilon m_\varepsilon(\varphi)-⨍_{D_\varepsilon}u^\varepsilon\star(-b\Delta w_\varepsilon)m_\varepsilon(\varphi)\\&-\int_{D_{R_\varepsilon}\setminus D_\varepsilon}u^\varepsilon\star(-\Delta w_\varepsilon)m_\varepsilon(\varphi)\Big\}=-4\pi\gamma_\varepsilon F_\varepsilon\star\int_\Omega G_{R_\varepsilon}(u^\varepsilon)m_\varepsilon(\varphi)\end{aligned}\tag{65}$$

Using (1), (50) and noticing that $\hat{F}_\varepsilon$ converges to $\hat{F}_{\alpha,b}$, then the proof is achieved by the Dominated Convergence Theorem because $|\hat{F}_\varepsilon|$ is also uniformly bounded by some holomorphic function.

■

The homogenized system is completed by the next result. It appears that the main macroscopic concentration satisfies a Volterra integro-differential equation (see [14]) and it defines straightly the macroscopic concentration associated to the suspension.

Theorem 4.9 *If u and v are the limits of (48)–(51), then they satisfy:*

$$\frac{\partial u}{\partial t}-\Delta u+4\pi\gamma(u+u\star h_{\alpha,b})=f-g-\frac{\partial w_0}{\partial t}\quad in\quad L^2(\Omega^T),\tag{66}$$

$$av-4\pi\gamma u\star F_{\alpha,b}=w_0+\int_0^t g(s)ds\quad in\quad L^2(\Omega^T),\tag{67}$$

$$u(0)=u_0\quad in\quad\Omega.\tag{68}$$

Remark 4.10 *From (67), it follows that in fact $v\in C^0([0,T];L^2(\Omega))$.*

Proof of Theorem 4.9. Let $\varphi\in\mathcal{D}(\Omega)$. First notice that assumption (40) on f^ε and the regularity of φ immediately yield

$$\int_\Omega\rho^\varepsilon f^\varepsilon\star w_\varepsilon m_\varepsilon(\varphi)\rightharpoonup\int_\Omega\int_0^t g(s)ds\varphi\quad\text{weakly in}\quad L^2(0,T).$$

From Proposition 4.6 and 4.7, we deduce that

$$\int_\Omega \left\{ \rho^\varepsilon \frac{\partial u^\varepsilon}{\partial t} \star w_\varepsilon - \operatorname{div}(k_\varepsilon \nabla u^\varepsilon) \star w_\varepsilon \right\} m_\varepsilon(\varphi) \rightharpoonup$$

$$\rightharpoonup a \int_\Omega v\varphi - \int_\Omega w_0 \varphi - 4\pi\gamma \int_\Omega F_{\alpha,b} \star u \, \varphi \quad \text{weakly in} \quad L^2(0,T),$$

from which we infer (67). Substituting the result into (58), we obtain (66). Notice that (67) also implies

$$av(0) = w_0(0)$$

and thus, (57) and (59) lead to (68). ■

Remark 4.11 *When $b \to +\infty$, then taking into account (29), we obtain*

$$\lim_{b\to+\infty} \hat{F}_{\alpha,b}(p) = \lim_{\alpha\to 0} \left(\frac{1}{p} - \frac{1}{p\left(1 + \frac{3a}{4\pi\gamma\alpha^2}(\alpha\sqrt{p}\coth(\alpha\sqrt{p}) - 1)\right)} \right) =$$

$$= \frac{1}{p + \frac{4\pi\gamma}{a}} =: \hat{F}_0(p),$$

and hence $F_0(t) = \exp(-4\pi\gamma t/a)$. Then (67) becomes

$$av = 4\pi\gamma F_0 \star u + w_0 + \int_0^t g(s)ds$$

and we derive the system:

$$\frac{\partial u}{\partial t} - \Delta u + 4\pi\gamma(u - v) = f - g - \frac{\partial w_0}{\partial t} - \frac{4\pi\gamma}{a}\left(w_0 + \int_0^t g(s)ds\right) \quad in \quad \Omega^T,$$

$$a\frac{\partial v}{\partial t} + 4\pi\gamma(v - u) = g + \frac{\partial w_0}{\partial t} + \frac{4\pi\gamma}{a}\left(w_0 + \int_0^t g(s)ds\right) \quad in \quad \Omega^T,$$

$$u(0) = u_0, \quad v(0) = v_0 \quad in \quad \Omega,$$

which is the homogenization result that we already presented in [6].

Proposition 4.12 *The limit problem (66)–(68) admits a unique solution $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$.*

Remark 4.13 *It follows that the convergences (48)–(51) hold for the whole sequence u^ε.*

Proof of Proposition 4.12. By the linearity of (66)–(68) it suffices to verify that $u \equiv 0$ is the unique solution of the homogeneous problem. Indeed, multiplying (66) by $u(t)$ and integrating over $\Omega \times (0,t)$, we get, using (68):

$$\frac{|u(t)|^2}{2} + \int_0^t |\nabla u|_\Omega^2 + 4\pi\gamma \int_0^t |u|_\Omega^2 = -4\pi\gamma \int_0^t \int_0^s h_{\alpha,b}(\tau)u(s-\tau)u(t)d\tau dt$$

from which we deduce

$$|u(t)|^2 \leq C \int_0^t \int_0^s |h_{\alpha,b}(\tau)||u(s-\tau)||u(t)|d\tau dt. \tag{69}$$

There results

$$|u(s)|_\Omega^2 \leq \sup_{0\leq\tau\leq t} |u(\tau)|_\Omega^2 = |u^2|_{L^\infty(0,t;L^1(\Omega))} < +\infty, \quad \forall s \in [0,t],$$

and consequently

$$|u(s)|_\Omega |u(r)|_\Omega \leq |u^2|_{L^\infty(0,s;L^1(\Omega))}, \quad \forall r \leq s,$$

which implies

$$|u(s)|_\Omega |u|_{L^\infty(0,s;L^2(\Omega)} \leq |u^2|_{L^\infty(0,s;L^1(\Omega))}, \forall s \leq t. \tag{70}$$

Now, notice that (69) also implies

$$|u(t)|_\Omega^2 \leq C \int_0^t \int_0^s |h_{\alpha,b}(\tau)||u(s)|_\Omega |u|_{L^\infty(0,s;L^2(\Omega))} d\tau ds.$$

Using (70) and setting $\eta(s) = \displaystyle\int_0^s |h(\tau)|d\tau$, there results

$$|u(t)|_\Omega^2 \leq C \int_0^t \eta(s)|u^2|_{L^\infty(0,s;L^1(\Omega))} ds, \quad \forall t \in [0,T[. \tag{71}$$

As for any $r \leq t$ we have

$$|u(r)^2|_\Omega = |u(r)|_\Omega^2 \leq C \int_0^r \eta(s)|u^2|_{L^\infty(0,s;L^1(\Omega))} ds \leq C \int_0^t \eta(s)|u^2|_{L^\infty(0,s;L^1(\Omega))} ds,$$

we find

$$|u^2|_{L^\infty(0,t;L^1(\Omega))} \leq \int_0^t \eta(s)|u^2|_{L^\infty(0,s;L^1(\Omega))} ds. \tag{72}$$

Applying Gronwall's Lemma, from (72) we obtain

$$|u^2|_{L^\infty(0,t;L^1(\Omega))} = 0 \quad \text{for any } t \in [0,T],$$

that is, $u \equiv 0$ in $L^\infty(0,T;L^2(\Omega))$, which achieves the proof. ■

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